Reconstructing the neutron-star equation of state from gravitational-wave observations

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Second generation gravitational-wave detectors

- Will reach design sensitivity around end of decade
- Sensitive to gravitational-waves between \( \sim 10 \) Hz and a few kHz
Stages of BNS coalescence

- Advanced LIGO sensitive to last few minutes of inspiral
- $\sim 10^4$ gravitational-wave cycles
Stages of BNS coalescence

- Early inspiral: Evolution depends on chirp mass $M = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$ and symmetric mass ratio $\eta = \frac{m_1 m_2}{(m_1 + m_2)^2}$.
Stages of BNS coalescence

- **Late inspiral**: EOS-dependent tidal interactions lead to phase shift of \( \sim 1 \) radian up to 400Hz.
Stages of BNS coalescence

- **Last 20-30 cycles**: Tidal interactions lead to phase shift of ~1 GW cycle.

400Hz up to merger
Stages of BNS coalescence

- **Post-merger**: Frequencies are a few kHz and depend sensitively on EOS
Stages of BNS coalescence

Fourier transform of waveform:

\[ \sqrt{S_n(f)} \text{ and } 2(f \cdot |h(f)|)^{1/2} \]

- Effectively point-particle
- Initial LIGO
- Advanced LIGO
- NS–NS EOS HB
- Tidal effects
- NS–NS merger
- AFTER NSNS merger

Waveforms from SACRA and WHISKY codes
(Credit: Jocelyn Read, arXiv:1306.4065)
Tidal interactions during inspiral

- Tidal field $\mathcal{E}_{ij}$ of each star induces quadrupole moment $Q_{ij}$ in other star

- Amount of deformation depends on the stiffness of the EOS via the tidal deformability $\lambda$

$$Q_{ij} = -\lambda(\text{EOS}, m) \mathcal{E}_{ij}$$
$$= -\Lambda(\text{EOS}, m)m^5 \mathcal{E}_{ij}$$

- Interaction increases binding energy

- Additional quadrupole moments increase gravitational radiation

$$\frac{dE}{dt} = -(1/5)\langle \tilde{Q}_{ij}^{\text{total}} \tilde{Q}_{ij}^{\text{total}} \rangle$$
Tidal interactions during inspiral

- Post-Newtonian approximation expands solution to Einstein equations in powers of speed of bodies and compactness of the system:

\[ x \equiv \left( \frac{GM\Omega}{c^3} \right)^{2/3} \sim \left( \frac{v}{c} \right)^2 \sim \frac{GM}{c^2d} \]

- Energy and gravitational-wave luminosity expansions:

\[ E = -\frac{1}{2} c^2 M \eta x \left[ 1 + e_{PP-PN}(x; \eta) + e_{Tidal}(x; \eta, \Lambda_1, \Lambda_2) \right] \quad \text{1PN–4PN} \]

\[ L = \frac{32}{5} c^5 G \eta^2 x^5 \left[ 1 + l_{PP-PN}(x; \eta) + l_{Tidal}(x; \eta, \Lambda_1, \Lambda_2) \right] \quad \text{1PN–3.5PN} \]

- Orbital evolution found with energy balance:

\[ \frac{dx}{dt} = \frac{dE}{dt} = \frac{dE}{dx} = \frac{-L}{dE/dx} \]

\[ \frac{d\phi}{dt} \equiv \Omega = \frac{c^3 x^{3/2}}{GM} \]

- Waveform is then:

\[ h_+ + ih_\times \propto \frac{\eta M}{d_L} x(t) e^{2i\phi(t)} \]
Tidal interactions during inspiral

- Tidal parameters encoded in phase evolution of waveform

\[
\tilde{h}(f) = \frac{A(\alpha, \delta, \iota, \psi)}{d_L} M^{5/6} f^{-7/6} e^{i\psi(f)}
\]

\[
\psi(f) = 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128(\pi M f)^{5/3}} \left[ 1 + \tilde{\psi}_{(\text{PP-PN})}(x; \eta) - \frac{39}{2} \tilde{\Lambda} x^5 + \left(-\frac{3115}{64} \tilde{\Lambda} + \frac{6595}{364} \sqrt{1 - 4\eta \delta \Lambda} \right)x^6 \right]
\]

\[
\tilde{x} = (\pi M f)^{2/3} \sim \left(\frac{v}{c}\right)^2
\]

\[
\tilde{\Lambda} = \frac{8}{13} \left[ (1 + 7\eta - 31\eta^2)(\Lambda_1 + \Lambda_2) + \sqrt{1 - 4\eta(1 + 9\eta - 11\eta^2)(\Lambda_1 - \Lambda_2)} \right]
\]

\[
\delta \tilde{\Lambda} = \frac{1}{2} \left[ \sqrt{1 - 4\eta} \left( 1 - \frac{13272}{1319} \eta + \frac{8944}{1319} \eta^2 \right)(\Lambda_1 + \Lambda_2) + \left( 1 - \frac{15910}{1319} \eta + \frac{32850}{1319} \eta^2 + \frac{3380}{1319} \eta^3 \right)(\Lambda_1 - \Lambda_2) \right]
\]
EOS fit

- One-to-one relation between EOS and radius-mass curves
- As well as between EOS and tidal deformability-mass curves
**EOS fit**

- Purely phenomenological EOS with 4 free parameters
- Methods apply to any EOS with free parameters

\[
p(\rho) = \begin{cases} 
K_1 \rho^{\Gamma_1}, & \rho_0 < \rho < \rho_1 \\
K_2 \rho^{\Gamma_2}, & \rho_1 < \rho < \rho_2 \\
K_3 \rho^{\Gamma_3}, & \rho > \rho_2 
\end{cases}
\]

\[\begin{align*}
\Gamma_1 & < \rho_1 < \Gamma_2 < \rho_2 < \Gamma_3 \\
\rho_1 & \text{ fixed} \\
\rho_2 & \text{ fixed} 
\end{align*}\]
Step 1: Estimate masses and tidal deformability

- Can estimate parameters of each BNS inspiral from Bayes’ Theorem:

\[
p(\theta|d_n) = \frac{p(\theta)p(d_n|\theta)}{p(d_n)}
\]

- \( \theta = \{d_L, \alpha, \delta, \psi, \nu, t_c, \phi_c, M, \eta, \Lambda, \delta \Lambda \} \)

- \( d_n \): data from nth BNS event
Step 1: Estimate masses and tidal deformability

- Can estimate parameters of each BNS inspiral from Bayes’ Theorem:

\[
p(\theta | d_n) = \frac{p(\theta)p(d_n | \theta)}{p(d_n)} = \frac{\text{Likelihood}}{\text{Evidence}}
\]

- Time series of stationary, Gaussian noise has the distribution

\[
p_n[n(t)] \propto e^{-(n,n)/2}
\]

\[
(a, b) = 4 \text{Re} \int_0^\infty \tilde{a}(f)\tilde{b}(f) \frac{S_n(f)}{S_n(f)} df
\]

- Likelihood of observing data \( d \) for gravitational wave model \( m(t; \tilde{\theta}) \) with parameters \( \tilde{\theta} \)

\[
p(d | \tilde{\theta}) \propto e^{-(d-m,d-m)/2}
\]

- where (data) = (noise) + (GW signal)
Step 1: Estimate masses and tidal deformability

- Can estimate parameters of each BNS inspiral from Bayes’ Theorem:

  \[
  p(\theta | d_n) = \frac{p(\theta)p(d_n | \theta)}{p(d_n)}
  \]

- Use Markov Chain Monte Carlo (MCMC) to sample posterior and marginalize over nuisance parameters

  \[
  p(M, \eta, \tilde{\Lambda} | d_n) = \int p(\theta | d_n) d\theta_{\text{nuisance}}
  \]
Step 1: Estimate masses and tidal deformability

3-detector LIGO-Virgo network with network SNR=20
Parameters estimated with LALInferenceMCMC

68% Credible region
95%
99.7%
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$\mathcal{O}(10^{-4})$

$\mathcal{O}(10^{-2})$

$\mathcal{O}(1)$

68% Credible region
95%
99.7%
True EOS
Step 2: Estimate EOS parameters

• Use Bayes’ theorem again to estimate masses and EOS parameters:

\[
\frac{\text{Prior \ Likelihood}}{\text{Evidence}} = \frac{p(\tilde{x})p(d_1 \ldots d_N | \tilde{x})}{p(d_1 \ldots d_N)}
\]

\[
\tilde{x} = \{ \log(p_1), \Gamma_1, \Gamma_2, \Gamma_3, M_1, \eta_1, \ldots, M_N, \eta_N \}\]
Step 2: Estimate EOS parameters

- Use Bayes’ theorem again to estimate masses and EOS parameters:

\[
p(\bar{x} | d_1 \ldots d_N) = \frac{p(\bar{x})p(d_1 \ldots d_N | \bar{x})}{p(d_1 \ldots d_N)}
\]

- Causality: Speed of sound must be less than the speed of light

\[v_s = \sqrt{\frac{dp}{d\epsilon}} < c\]

- Maximum mass: EOS must support observed stars with masses greater than \(1.93 M_\odot\)
Step 2: Estimate EOS parameters

- Use Bayes’ theorem again to estimate masses and EOS parameters:

\[
P(x|d_1 \ldots d_N) = \frac{p(x)p(d_1 \ldots d_N|x)}{p(d_1 \ldots d_N)}
\]

- Total likelihood is product of likelihoods for each independent event

- Rewritten in terms of the EOS parameters instead of tidal deformability

- Marginalized posterior for single event

\[
p(d_1, \ldots, d_N|x) = \prod_{n=1}^{N} p(M_n, \eta_n, \tilde{\Lambda}_n|d_n) | \tilde{\Lambda}_n = \tilde{\Lambda}(M_n, \eta_n, \text{EOS})
\]

- EOS parameters found from MCMC simulation for 4+2N parameters by marginalizing over the 2N mass parameters
Simulating a population of BNS events

- Sampled a year of data using the standard “realistic” event rate
  - ~40 BNS events/year for single detector with SNR>8
- Masses sampled uniformly in $[1.2M_\odot, 1.6M_\odot]$
- Chose MPA1 to be “true” EOS when calculating tidal parameters for these events
- Injected waveforms into simulated noise for the 3-detector LIGO-Virgo network
Results for 1 year of data
Results for 1 year of data
Higher mass NS observations

- Black widow pulsars may have particularly high masses, but large systematic uncertainties
  - PSR B1957+20: $2.40 \pm 0.12 M_\odot$
  - PSR J1311-3430: $2.68 \pm 0.14 M_\odot$
- Higher mass NS observations improve the measurability at higher masses
Simulated BNS populations where all the masses were fixed at $1.0M_\odot$, $1.4M_\odot$, or $1.8M_\odot$.

Errors are smallest near the masses of the simulated population.

Can still measure NS properties at other masses due to prior constraints on the equation of state.
Other EOS models
Systematic errors

- Several ways to calculate waveform phase from energy and luminosity expressions
- Phase difference between 3PN and 3.5PN as big as tidal effect
- Phase difference between TaylorT1 and TaylorT4 as big as tidal effect

Hinderer et al. arXiv:0911.3535
Systematic errors

- Injected TaylorF2, TaylorT1, TaylorT4 waveform models
- Used TaylorF2 as template
Systematic errors

- Several ways to improve the waveform model
  - Effective one body waveforms
  - Reproduce BBH waveforms to high accuracy
  - Recent comparisons with BNS simulations are promising
- Numerical simulations are the only solution once NSs are in contact

\[ m_1 = m_2 = 1.35M_\odot, \text{ EOS=SLy} \]

Sebastiano Bernuzzi et al.  
Conclusions

• The BNS inspiral waveform provides detailed EOS information

• 1 year of data will be sufficient to measure (statistical error):
  • Pressure to less than a factor of 2
  • Radius to +/- 1 km

• Systematic errors from inexact waveform templates will be primary difficulty in measuring the EOS
  • Will be reduced in the near future with improved waveform models
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Thank you